Notiztitel 26.09.2008

David Gross

TU Braunschweig

The stabilizer formalism.

(Or: Why Ψ sucks)

I) Why state vectors are bad

Pessimist QM doesn't work

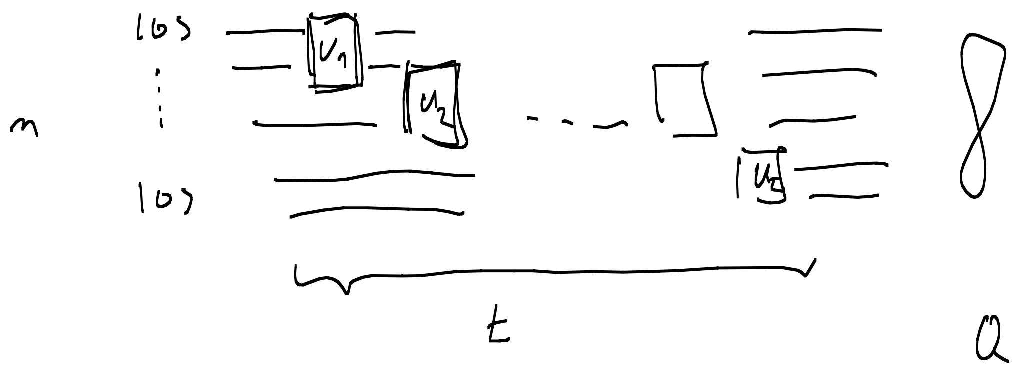
$\uparrow \quad \downarrow \quad \cdots \quad \uparrow$

$1 \quad 2 \quad \cdots \quad 100$ $\dim \mathcal{H} = 2^{100}$

$|\Psi\rangle$ There's not enough paper in the universe.

Optimist

16 parameters



$|\Psi(U_1,\dots,U_t)\rangle$

QM has a chance. Physical states have fewer params.

$16 \cdot \underline{n(n-1)} \cdot t \ll 2^n$

P: OK, but take an observable

- $A_1 \otimes A_2 \cdots \otimes A_n$
- $|\Psi(U_1,\dots,U_t)\rangle$

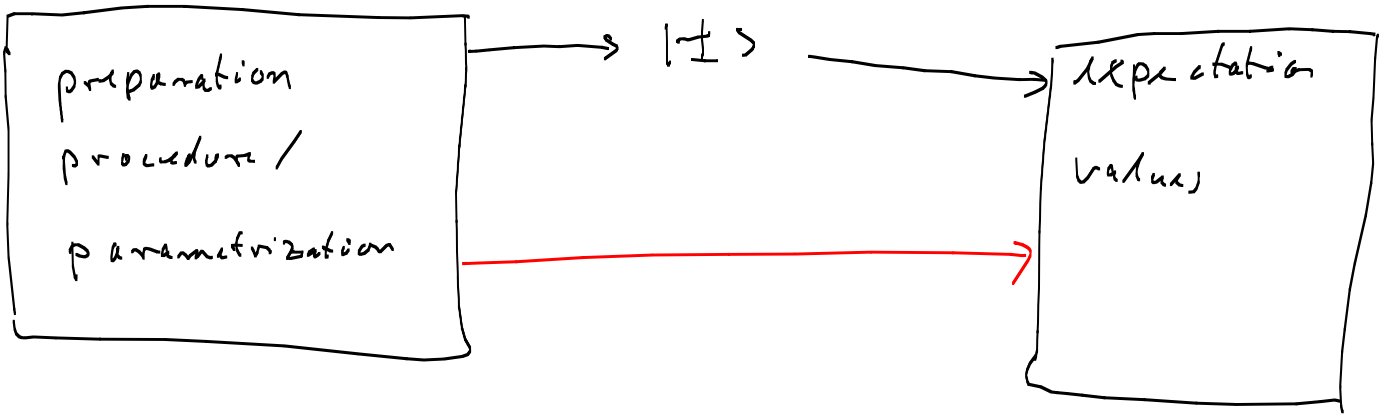
How to get

$$f(U_1,\ldots,U_t) = \langle \Psi(U_1,\ldots,U_t) | A_1 \otimes \cdots \otimes A_n | \Psi(U_1,\ldots,U_t) \rangle$$

Not enough time in the universe.

Q: I will identify a subset of states in $(\mathbb{C}^2)^{\otimes n}$ such that

- have few param.
- possible to efficiently compute some expectation values
- interesting.



II Stabilizers

a) Pauli matrices

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sigma_0$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1$$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_2$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3$$

b) composition

$$X \cdot Y = i Z$$

$$Z X = i Y$$

$$Y Z = i X$$

c) square to $\mathbb{1}$

$$X^2 = \mathbb{1}$$

$$Y^2 = \mathbb{1}$$

$$Z^2 = \mathbb{1}$$

d) anti commute

$$X Y = - Y X$$

$$\sigma_i \sigma_j = - \sigma_j \sigma_i$$

$i \neq j \qquad i, j = 1, 2, 3$

e)

$$\mathrm{tr}\, X = \mathrm{tr}\, Y = \mathrm{tr}\, Z = 0$$

$$\mathrm{tr}\, \mathbb{1} = 2.$$

4) n-qubit Pauli operators

$$\mathcal{P}^{(n)} = \{ c\,\sigma_{i_1} \otimes \sigma_{i_2} \otimes \cdots \otimes \sigma_{i_n} \mid i_1, \ldots, i_n = 0,1,2,3$$

$$c = +1, -1, +i, -i \}$$

Ex.: $\mathcal{P}^{(2)} \ni \mathbb{1} \otimes X,\ Z \otimes Z,\ -i\, Y \otimes X \ldots$

O's idea: Look for interesting states among the eigenvectors of Pauli matrices.

Generalize

Let $g_1, \ldots, g_k \in \mathcal{P}^{(n)}$

such that

- they all commute: $g_i g_j = g_j g_i$
- $g^2 = \mathbb{1} \quad \forall g \in S$
- S stabilizer group

$\langle g_1, \ldots, g_k \rangle$

Def.: The stabilizer space V_S is the set of vectors $|\psi\rangle$

such that

$$g\,|\Psi\rangle = |\Psi\rangle$$

for all $g \in S$.

Q: • When is V_S interesting?

• Given S, how do I get it?

$\rightsquigarrow$ analyze these gr.

I) $S = \langle g_1, \dots, g_k \rangle$

$$g_1\, g_2\, g_4\, g_2$$

$$= g_1\, \underbrace{g_2\, g_2}_{\mathbb{1}}\, g_4 = g_1\, g_4$$

$$= g_1^{1} \cdot g_2^{0} \cdot g_3^{0} \cdot g_4^{1} \cdot g_5^{0} \cdots g_k^{0}$$

$S \ni g \rightsquigarrow g_1^{x_1} \cdot g_2^{x_2} \cdot \ldots \cdot g_k^{x_k}$ $\quad x_i = 0/1$

$2 \cdot 2 \cdots 2 = 2^k$

$$\boxed{|S| = 2^k.}$$

Problem: $S \longrightarrow$ stabilizer space?

solved by looking at

$$P_S = \frac{1}{2^k} \sum_{g \in S} g .$$

1) $$\operatorname{tr} P_S = \frac{1}{2^k} \sum_{g \in S} \underbrace{\operatorname{tr} g}$$

$$\operatorname{tr} \sigma_{i_1} \otimes \dots \otimes \sigma_{i_n}$$

$$= (\operatorname{tr} \sigma_{i_1}) \cdot \ldots \cdot (\operatorname{tr} \sigma_{i_n})$$

$$= \begin{cases} 0 & \text{if } g \neq 1 \otimes \dots \otimes 1 \\ 2^n & \text{if } g = 1 \otimes \dots \otimes 1 . \end{cases}$$

$$= \frac{2^n}{2^k} = \underline{2^{n-k}} .$$

2) $$\underline{P_S \cdot P_S} = \frac{1}{2^k} \cdot \frac{1}{2^k} \sum_{g \in S} g \sum_{h \in S} h$$

$$= \frac{1}{2^k} \cdot \frac{1}{2^k} \sum_g \underbrace{\left(\sum_h g h \right)}_{= \sum_h h}$$

$$= \frac{1}{2^k} \sum_{g \in S} P_S = \underline{P_S} .$$

$\rightsquigarrow$ P_S is a projection operator

$\rightsquigarrow$ it projects on $\mathrm{tr}\, P_S = 2^{n-k}$

dimensional space.

<u>Seen:</u> take n Pauli operators

$g_1, \dots, g_n$

$\rightsquigarrow$ $S = \langle g_1, \dots, g_n \rangle$

$\rightsquigarrow$ $P = \frac{1}{2^n} \sum_{g \in S} g$

$\rightsquigarrow$ $P = \underbrace{|\Psi\rangle\langle\Psi|}_{\text{stabilizer state,}}$

<u>Observables</u>

I have $|\Psi\rangle$, a stabilizer state.

Have observable, e.g. $h = Z \otimes \dots \otimes Z$

$$\mathrm{tr}\left[|\Psi\rangle\langle\Psi| \; Z \otimes \dots \otimes Z \right]$$

$$= \mathrm{tr}\left[P_S \quad \underbrace{Z \otimes \dots \otimes Z}_{} \right]$$

$$= \frac{1}{2^n} \sum_{g \in S} \underbrace{\mathrm{tr}[\, g \cdot h \,]}_{= \begin{cases} 2^n & g \cdot h = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \\ 0 & \text{else} \end{cases}}$$

$$= \begin{cases} 2^n & g = h \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} 1 & h \in S \\ 0 & \text{else.} \end{cases}$$

Fantastic!